2.1 Transforming Linear Functions

Before we begin looking at transforming linear functions, let's take a moment to review how to graph linear equations using slope-intercept form. This will help us because the easiest way to think of transformations is graphically.

Slope-Intercept Form

Any linear equation can be written in the form $y = mx + b$ where m is the slope and b is the y-intercept. Sometimes the equation we need to graph will already be in slope-intercept form, but if it's not, we'll need to rearrange the equation to get it into slope-intercept form. Take a look at the following equations:

Example 1

$$y = 2x - 1$$

This equation is already in slope-intercept form. Nothing needs to be done.

Example 2

$$2x + y = 7$$

This equation is not in slope-intercept form. We need to subtract $2x$ from both sides to get the y by itself.

$$2x - 2x + y = 7 - 2x$$

$$y = -2x + 7$$

Example 3

$$3x - 2y = 4$$

This example is also not in slope-intercept form. We'll first subtract $3x$, but then notice that we'll be left with a $-2y$. Be careful because that negative sign is important. Next divide by -2 to get y by itself.

$$3x - 3x - 2y = 4 - 3x$$

$$-2y = -3x + 4$$

$$\frac{-2y}{-2} = \frac{-3x + 4}{-2}$$

$$y = \frac{3}{2}x - 2$$

Example 4

$$-4x + 2y = 8$$

This is not in slope-intercept form. We'll first need to get rid of the $-4x$ by adding $4x$ and then we'll have to get rid of the times by 2 by dividing by 2. That will get y by itself.

$$-4x + 4x + 2y = 8 + 4x$$

$$2y = 4x + 8$$

$$\frac{2y}{2} = \frac{4x + 8}{2}$$

$$y = 2x + 4$$

So, step one in graphing is to get the equation in slope-intercept form.

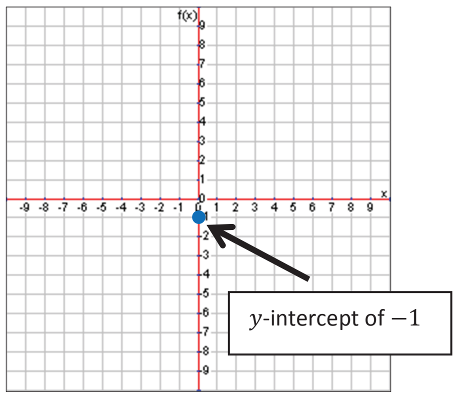
The y-Intercept and the Slope

Once you have an equation in slope-intercept form, start by graphing the y-intercept on the coordinate plane. From the y-intercept, move the rise and run of the slope to plot another point. Finally, draw the line that connects the two points. Let's use our previous equations to graph step-by-step.

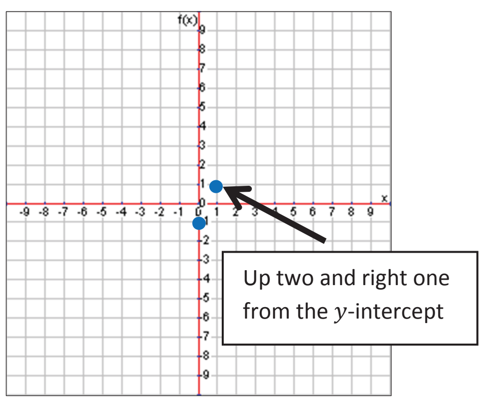
Example 1

$$y = 2x - 1$$

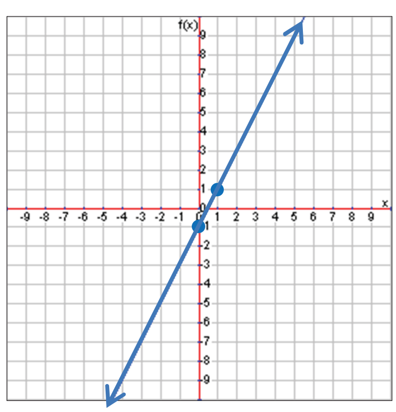
The y-intercept is -1, so we plot a point at -1 on the y-axis to start.



Next we know the slope is 2 which means a rise of 2 and a run of 1.
So we'll move up two and right one to plot the next point.

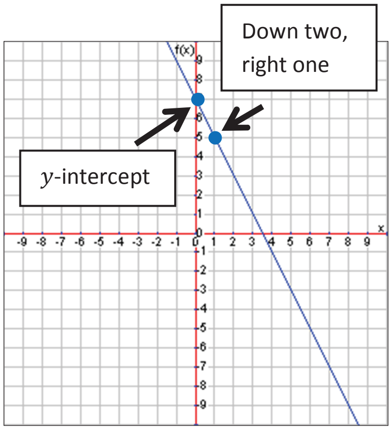


Finally, connect the dots with a line.



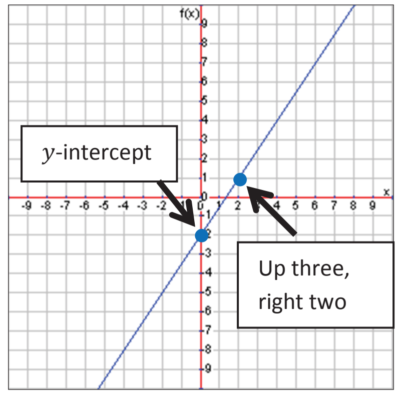
Example 2

$y = -2x + 7$



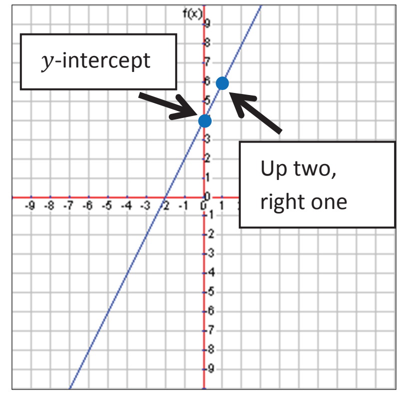
Example 3

$y = \frac{3}{2}x - 2$



Example 4

$y = 2x + 4$



Transformations

Now let's flip back to function notation to talk about transformations. So consider the basic form $f(x) = mx + b$ of a linear function. For any function $f(x)$, not just linear, there are four ways that we can transform it. We can add something to the input or the output, or we can multiply something by the input or the output. Let's take each one individually.

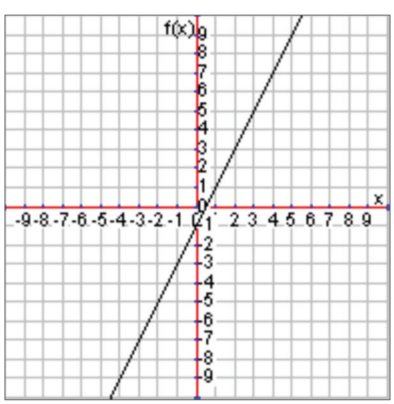
Adding to the output

In function notation, this looks like $f(x) + k$. (Notice that k can be negative which would mean that we are in essence subtracting.) What this notation means is that we go ahead and calculate the value of the function as normal and then add to it. So let's take a look at $f(x) = 2x - 1$ and transform it by looking at $g(x) = f(x) + 5$. What would happen?

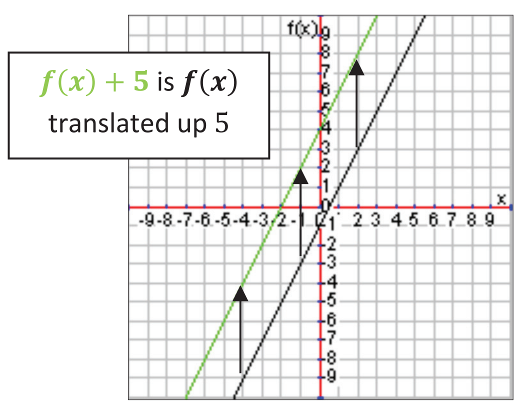
$$f(x) + 5 = 2x - 1 + 5 = 2x + 4$$

Now the y-intercept is now 4 instead of -1 which means that the function has moved up 5 units. So by adding five to the output, we simply moved the function up five.

Original: $f(x) = 2x - 1$



Transformed: $f(x) + 5 = 2x + 4$



Thus, if we subtracted three from the output, $f(x) - 3$, that would move the function down three units.

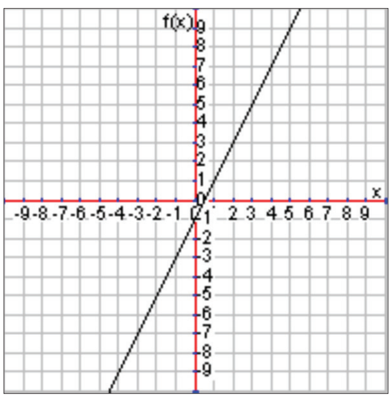
Adding to the input

In function notation, this looks like $f(x+k)$. (Notice that k can be negative which would mean that we are in essence subtracting.) What this notation means is that we first add to our input then use that value to calculate the value of the function as normal. So let's take a look at $f(x) = 2x - 1$ and transform it by looking at $f(x+5)$. What would happen?

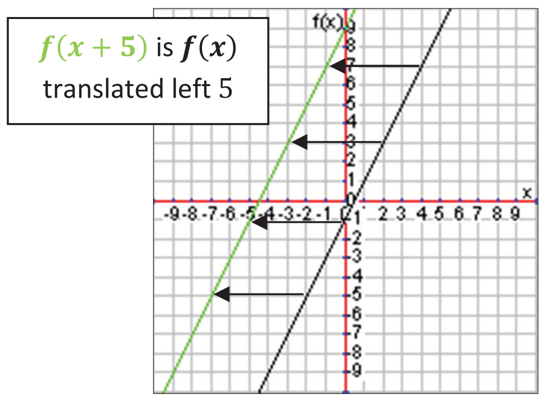
$$f(x+5) = 2(x+5) - 1 = 2x + 10 - 1 = 2x + 9$$

It looks like this translated (moved) the function up by ten, which is true. However, there is better way to think about this. Notice that an input of 0 in $f(x+5)$ now looks like an input of 5 in $f(x)$. In other words, what was an input of five is now an input of zero. So a better way to think about this is that it translated the function five units to the left.

Original: $f(x) = 2x - 1$



Transformed: $f(x+5) = 2x = 9$

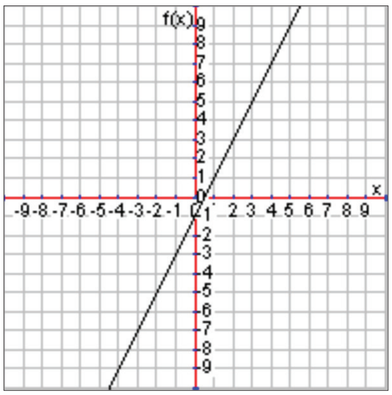


So if we had $f(x+2)$, it would translate left two. If we had $f(x-4)$, it would translate right four. One way to think about it is to think about what x value would make the input zero. In $f(x+2)$, the x value of -2 would make the input zero. Negative two in the x direction on a graph is left two.

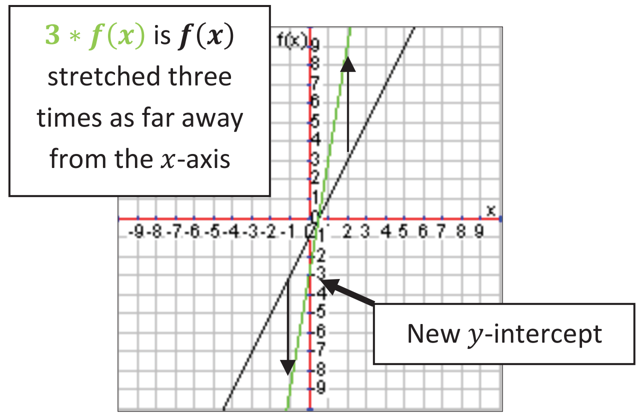
Multiplying by the output

In function notation, this looks like $k * f(x)$. We first calculate the function value with the regular input and then we multiply it by some value. Let's imagine that $k = 3$. This means that the output of the transformed function will be three times as big as the original function. We basically have stretched the graph away from the x-axis. Consider the function $f(x) = 2x - 1$ and the transformation $3 * f(x) = 3(2x - 1) = 6x - 3$.

Original: $f(x) = 2x - 1$



Transformed: $3 * f(x) = 6x - 3$

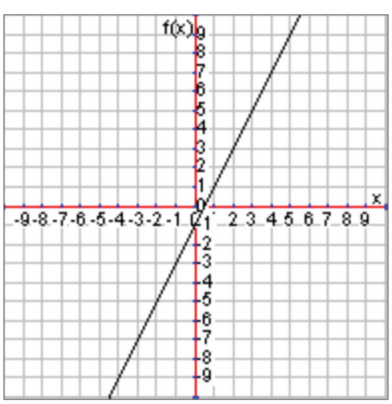


Notice that the x-intercept stayed in place, but the y-intercept changed. Why would that happen? Since the value of the function at the x-intercept is zero, zero times anything is still zero. So that point doesn't move. What would happen if k were a fraction between zero and one? What would happen if k were negative?

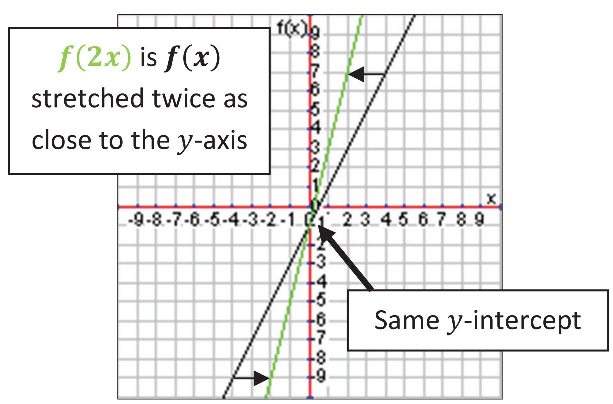
Multiplying by the input

In function notation, this looks like $f(k * x)$. We multiply the input by some value and then use that to calculate the value of the transformed function. Let's imagine that $k = 2$. This means that what was an input of one is now acting like an input of two, so the output previously associated with the input of two is now associated with the input of one. What was an input of five is now acting like an input of ten, so the output previously associated with the input of five is now associated with the input of five. We basically have stretched the graph closer to the y-axis this time. Consider the function $f(x) = 2x - 1$ and the transformation $f(2x) = 2(2x) - 1 = 4x - 2$.

Original: $f(x) = 2x - 1$



Transformed: $f(2x) = 4x - 2$

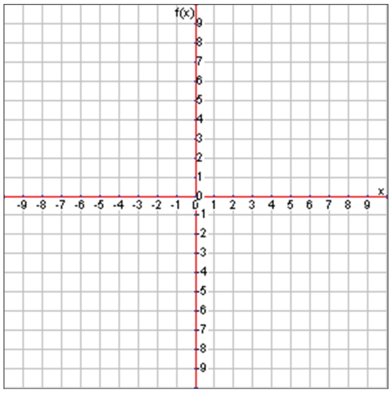


Notice that the y-intercept stayed in place, but the x-intercept changed. Why would that happen? Since the input at the y-intercept is zero, zero times anything is still zero. So that point doesn't move. What would happen if k were a fraction between zero and one? What would happen if k were negative?

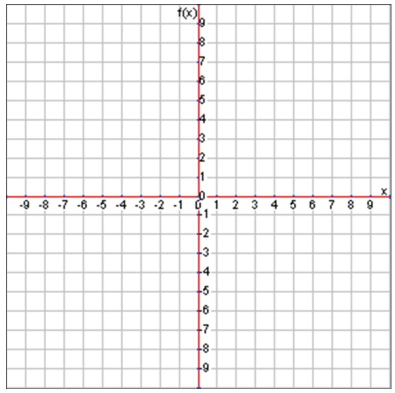
Lesson 2.1

Graph the following linear equations using slope-intercept form.

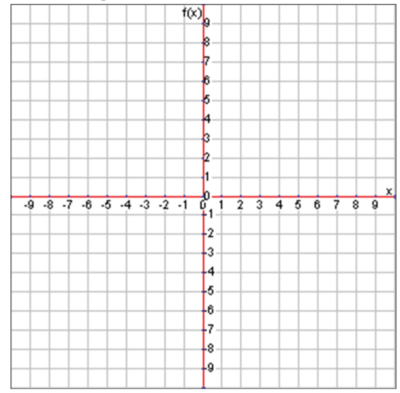
1. $y = 2x + 1$



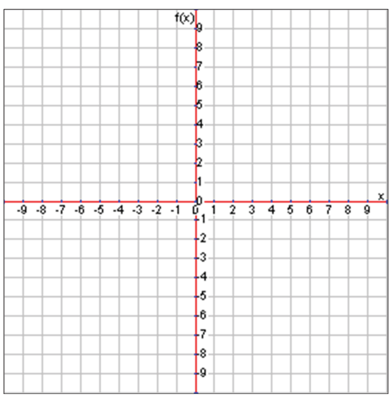
2. $y = 3x - 4$



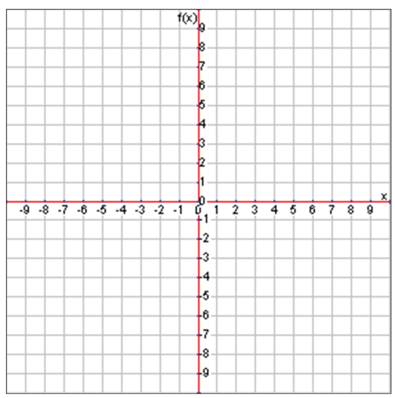
3. $y = \frac{2}{3}x + 5$



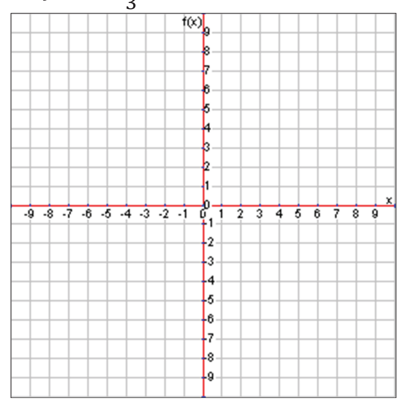
4. $y = 7$



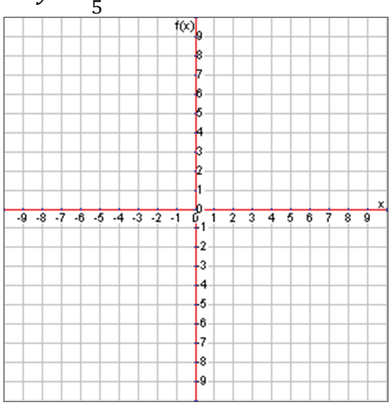
5. $y = -3x - 2$



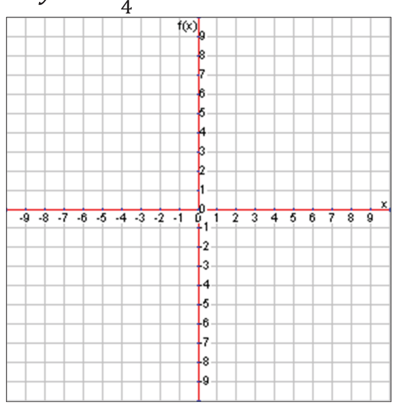
6. $y = -\frac{1}{3}x + 5$



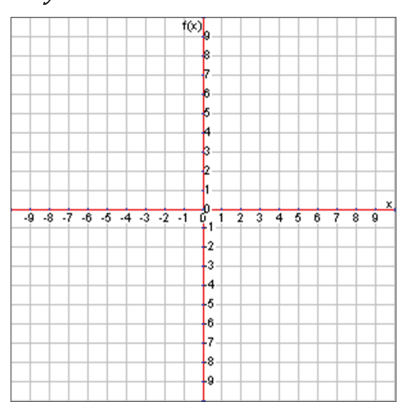
7. $y = \frac{2}{5}x - 2$



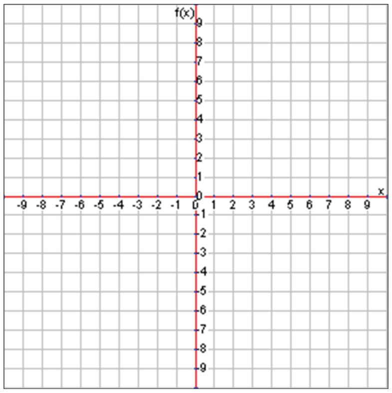
8. $y = -\frac{3}{4}x - 1$



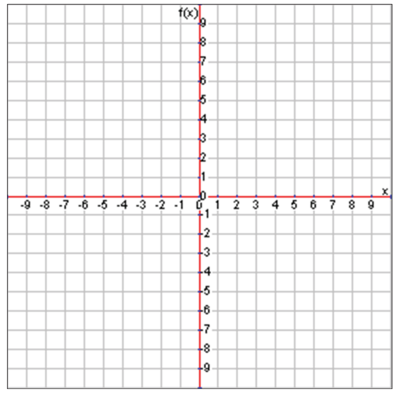
9. $y = -4$



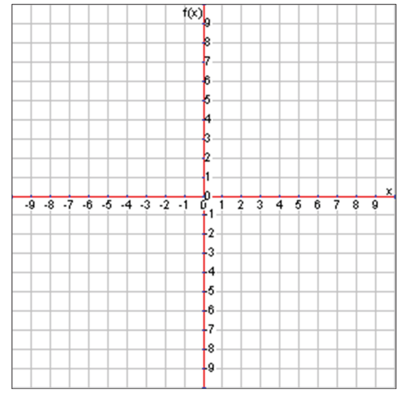
10. $2x + y = 2$



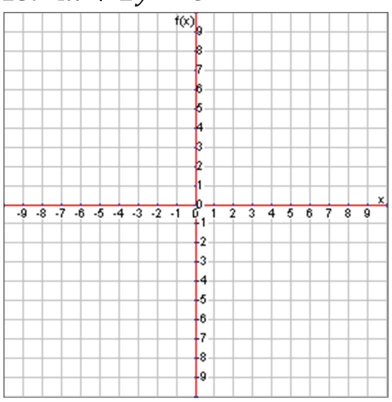
11. $-3x + y = 4$



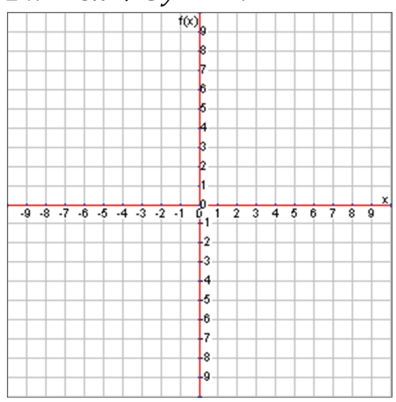
12. $4x + y = -5$



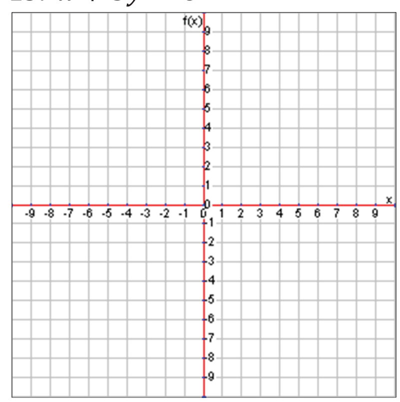
13. $4x + 2y = 6$



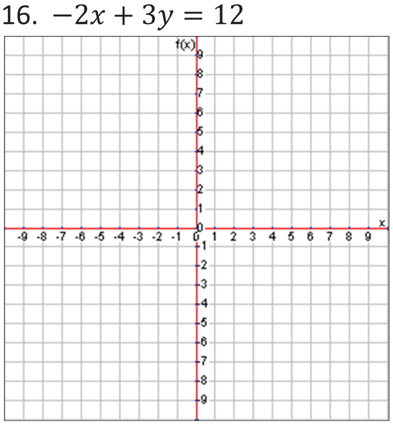
14. $-6x + 3y = -9$



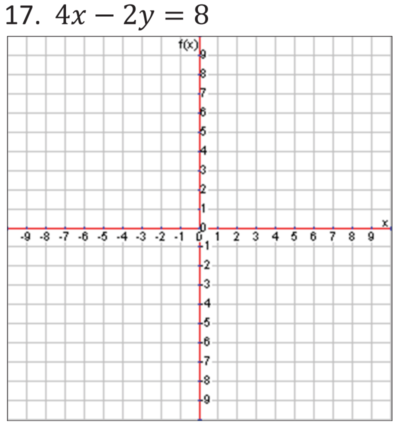
15. $x + 3y = 6$



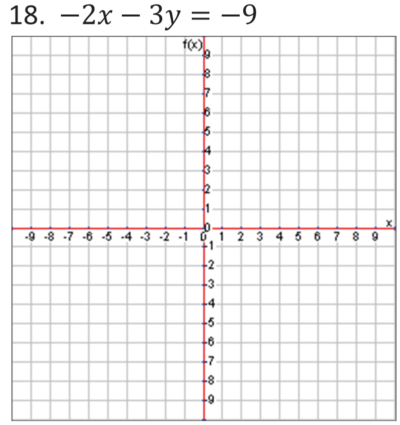
16. $-2x + 3y = 12$



17. $4x - 2y = 8$



18. $-2x - 3y = -9$



Use the given functions to describe the listed transformations. Then write the equation of the transformed function.

$f(x) = 3x - 2$ $\qquad g(x) = -2x + 4$ $\qquad h(x) = 5$

19. $f(2x)$

20. $2 * f(x)$

21. $f(x) - 2$

22. $f(x + 2)$

23. $g(-x)$

24. $\frac{1}{2} * g(x)$

25. $g(x) + 5$

26. $g(x - 5)$

27. $f(\frac{1}{2}x)$

28. $-3 * g(x)$

29. $h(x) - 4$

30. $h(x + 6)$

31. $h(2x)$

32. $2 * h(x)$

33. $f(x) - 3$